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Minimization of equilibrium problems, variational inequality problems and fixed point problems

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Abstract In this paper, we devote to find the solution of the following quadratic minimization problem

$$\min_{x \in \Omega} \|x\|^2,$$

where Ω is the intersection set of the solution set of some equilibrium problem, the fixed points set of a nonexpansive mapping and the solution set of some variational inequality. In order to solve the above minimization problem, we first construct an implicit algorithm by using the projection method. Further, we suggest an explicit algorithm by discretizing this implicit algorithm. Finally, we prove that the proposed implicit and explicit algorithms converge strongly to a solution of the above minimization problem.

Keywords Equilibrium problem · Minimization problem · Variational inequality · Monotone mapping · Fixed point

Mathematics Subject Classification (2000) 49J40 · 47J20 · 47H09 · 65J15

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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $A : C \to H$ is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is clear that any α-inverse-strongly monotone mapping is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. A mapping $S : C \to C$ is said to be *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Denote the set of fixed points of S by $F(S)$.

Let $B : C \to H$ be a nonlinear mapping and $F : C \times C \to R$ be a bifunction. Now we concern the following equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Bz, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of (1.1) is denoted by $EP(F, B)$. If $B = 0$, then (1.1) reduces to the following equilibrium problem of finding $z \in C$ such that

$$F(z, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solution set of (1.2) is denoted by $EP(F)$. If $F = 0$, then (1.1) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Bz, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The solution set of variational inequality (1.3) is denoted by $VI(C, B)$.

Equilibrium problems which were introduced by Blum and Oettli [1] in 1994 have had a great impact and influence in pure and applied sciences. It has been shown that the equilibrium problems theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization. Equilibrium problems include variational inequalities, fixed point, Nash equilibrium and game theory as special cases. The equilibrium problems and the variational inequality problems have been investigated by many authors. Please see [2–30] and the references therein. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others.

In this paper, we devote to find the solution of the following quadratic minimization problem

$$\min_{x\in\Omega} \|x\|^2,$$

where Ω is the intersection set of the solution set of some equilibrium problem, the fixed points set of a nonexpansive mapping and the solution set of some variational inequality. In order to solve the above minimization problem, we first construct an implicit algorithm by using the projection method. Further, we suggest an explicit algorithm by discretizing this implicit algorithm. Finally, we prove that the proposed implicit and explicit algorithms converge strongly to a solution of the above minimization problem.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Throughout this paper, we assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

(H1) $F(x, x) = 0$ for all $x \in C$;
(H2) F is monotone, i.e., $F(x, y) + F(y, x) \le 0$ for all $x, y \in C$;
(H3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \le F(x, y)$;
(H4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C(x) \Leftrightarrow \langle u - x, u - y\rangle \le 0, \quad \forall y \in C. \tag{2.1}$$

We need the following lemmas for proving our main results.

Lemma 2.1 ([8]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction which satisfies conditions* (H1)–(H4). *Let $\mu > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{\mu}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

Further, if $T_\mu(x) = \{z \in C : F(z, y) + \frac{1}{\mu}\langle y - z, z - x\rangle \ge 0 \text{ for all } y \in C\}$, then the following hold:

(a) *T_μ is single-valued and T_μ is firmly nonexpansive, i.e., for any $x, y \in C$, $\|T_\mu x - T_\mu y\|^2 \le \langle T_\mu x - T_\mu y, x - y\rangle$;*
(b) *$EP(F)$ is closed and convex and $EP(F) = F(T_\mu)$.*

Lemma 2.2 ([4]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone and $\lambda > 0$ be a constant. Then, we have*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \le \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2, \quad \forall x, y \in H.$$

In particular, if $0 \le \lambda \le 2\alpha$, then $I - \lambda A$ is nonexpansive.

Lemma 2.3 ([18]) *Let $\{x_n\}$ and $\{v_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)v_n + \beta_n x_n$ for all $n \ge 0$ and $\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty} \|v_n - x_n\| = 0$.*

Lemma 2.4 ([21]) *Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - S)x_n \to y$ strongly, then $(I - S)x^* = y$.*

Lemma 2.5 ([20]) *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section we will introduce two algorithms (one implicit and one explicit) for finding the minimum norm element x^* of $\Omega := EP(F, B) \cap VI(C, A) \cap F(S)$. Namely, we want to find a point x^* which solves the following minimization problem:

$$\|x^*\|^2 = \min_{x\in\Omega} \|x\|^2. \tag{3.1}$$

Let $S : C \to C$ be a nonexpansive mapping and $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone mappings, respectively. Let $F : C \times C \to R$ be a bifunction which satisfies conditions (H1)–(H4). In order to solve the quadratic minimization problem (3.1), we first construct the following implicit algorithm by using the projection method

$$x_t = SP_C[(1-t)P_C(I-\lambda A)T_\mu(I-\mu B)x_t], \quad \forall t \in (0, 1), \tag{3.2}$$

where T_μ is defined as Lemma 2.1 and λ, μ are two constants such that $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$. We will show that the net $\{x_t\}$ defined by (3.2) converges to a solution of the minimization problem (3.1). First, we show that the net $\{x_t\}$ is well-defined. As matter of fact, for each $t \in (0, 1)$, we consider the following mapping W_t given by

$$W_t x = SP_C[(1-t)P_C(I-\lambda A)T_\mu(I-\mu B)x], \quad \forall x \in C.$$

Since the mappings S, P_C, $I-\lambda A$, T_μ and $I-\mu B$ are nonexpansive, then we can check easily that $\|W_t x - W_t y\| \leq (1-t)\|x-y\|$ which implies that W_t is a contraction. Using the Banach contraction principle, there exists a unique fixed point x_t of W_t in C, i.e., $x_t = W_t x_t$ which is exactly (3.2).

Next we show the first main result of the present paper.

Theorem 3.1 *Suppose* $\Omega \neq \emptyset$. *Then the net* $\{x_t\}$ *generated by the implicit method* (3.2) *converges in norm, as* $t \to 0$, *to a solution of the quadratic minimization problem* (3.1).

Proof First, we prove that $\{x_t\}$ is bounded. Set $u_t = T_\mu(I-\mu B)x_t$ and $y_t = P_C(I-\lambda A)u_t$ for all $t \in (0, 1)$. Take $z \in \Omega$. It is clear that $z = T_\mu(z-\mu Bz) = P_C(z-\lambda Az)$. Since T_μ is nonexpansive and A, B are α-inverse-strongly monotone and β-inverse-strongly monotone, we have from Lemma 2.2 that

$$\begin{aligned}\|u_t - z\|^2 &= \|T_\mu(x_t-\mu Bx_t) - T_\mu(z-\mu Bz)\|^2\\ &\leq \|x_t-\mu Bx_t-(z-\mu Bz)\|^2\\ &\leq \|x_t-z\|^2 + \mu(\mu-2\beta)\|Bx_t-Bz\|^2,\end{aligned}$$

and

$$\begin{aligned}\|y_t - z\|^2 &= \|P_C(u_t - \lambda A u_t) - P_C(z - \lambda A z)\|^2 \\ &\le \|u_t - \lambda A u_t - (z - \lambda A z)\|^2 \\ &\le \|u_t - z\|^2 + \lambda(\lambda - 2\alpha)\|A u_t - A z\|^2 \\ &\le \|x_t - z\|^2 + \lambda(\lambda - 2\alpha)\|A u_t - A z\|^2 + \mu(\mu - 2\beta)\|B x_t - B z\|^2 \\ &\le \|x_t - z\|^2. \end{aligned} \tag{3.3}$$

So, we have that

$$\|y_t - z\| \le \|u_t - z\| \le \|x_t - z\|.$$

It follows from (3.2) that

$$\begin{aligned}\|x_t - z\| &= \|S P_C[(1-t)y_t] - S P_C z\| \\ &\le \|(1-t)(y_t - z) - tz\| \\ &\le (1-t)\|y_t - z\| + t\|z\| \\ &\le (1-t)\|x_t - z\| + t\|z\|,\end{aligned}$$

that is,

$$\|x_t - z\| \le \|z\|.$$

So, $\{x_t\}$ is bounded. Hence $\{u_t\}$ and $\{y_t\}$ are also bounded. Now we can choose a constant $M > 0$ such that

$$\sup_t \left\{2\|z\|\|y_t - z\| + \|z\|^2, 2\|u_t - y_t\|, 2\|x_t - u_t\|, \|y_t\|^2\right\} \le M.$$

From (3.2) and (3.3), we have

$$\begin{aligned}\|x_t - z\|^2 &\le \|(1-t)(y_t - z) - tz\|^2 \\ &= (1-t)^2\|y_t - z\|^2 - 2t(1-t)\langle z, y_t - z\rangle + t^2\|z\|^2 \\ &\le \|y_t - z\|^2 + tM \\ &\le \|x_t - z\|^2 + \lambda(\lambda - 2\alpha)\|A u_t - A z\|^2 \\ &\quad + \mu(\mu - 2\beta)\|B x_t - B z\|^2 + tM\end{aligned} \tag{3.4}$$

that is,

$$\lambda(2\alpha - \lambda)\|A u_t - A z\|^2 + \mu(2\beta - \mu)\|B x_t - B z\|^2 \le tM \to 0.$$

Since $\lambda(2\alpha - \lambda) > 0$ and $\mu(2\beta - \mu) > 0$, we derive

$$\lim_{t\to 0}\|A u_t - A z\| = \lim_{t\to 0}\|B x_t - B z\| = 0. \tag{3.5}$$

From Lemma 2.1 and 2.2 and (3.2), we obtain

$$\begin{aligned}\|u_t - z\|^2 &= \|T_\mu(x_t - \mu B x_t) - T_\mu(z - \mu B z)\|^2 \\ &\le \langle (x_t - \mu B x_t) - (z - \mu B z), u_t - z\rangle \\ &= \frac{1}{2}\Big(\|(x_t - \mu B x_t) - (z - \mu B z)\|^2 + \|u_t - z\|^2 \\ &\quad - \|(x_t - z) - \mu(B x_t - B z) - (u_t - z)\|^2\Big)\end{aligned}$$

$$\le \frac{1}{2}\left(\|x_t - z\|^2 + \|u_t - z\|^2 - \|(x_t - u_t) - \mu(Bx_t - Bz)\|^2\right)$$
$$= \frac{1}{2}\left(\|x_t - z\|^2 + \|u_t - z\|^2 - \|x_t - u_t\|^2 + 2\mu\langle x_t - u_t, Bx_t - Bz\rangle - \mu^2\|Bx_t - Bz\|^2\right),$$

and

$$\begin{aligned}\|y_t - z\|^2 &= \|P_C(u_t - \lambda Au_t) - P_C(z - \lambda Az)\|^2\\ &\le \langle (u_t - \lambda Au_t) - (z - \lambda Az), y_t - z\rangle\\ &= \frac{1}{2}\left(\|(u_t - \lambda Au_t) - (z - \lambda Az)\|^2 + \|y_t - z\|^2\right.\\ &\quad \left.-\|(u_t - \lambda Au_t) - (z - \lambda Az) - (y_t - z)\|^2\right)\\ &\le \frac{1}{2}\left(\|u_t - z\|^2 + \|y_t - z\|^2 - \|(u_t - y_t) - \lambda(Au_t - Az)\|^2\right)\\ &= \frac{1}{2}\left(\|u_t - z\|^2 + \|y_t - z\|^2 - \|u_t - y_t\|^2\right.\\ &\quad \left.+ 2\lambda\langle u_t - y_t, Au_t - Az\rangle - \lambda^2\|Au_t - Az\|^2\right).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|u_t - z\|^2 &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + 2\mu\langle x_t - u_t, Bx_t - Bz\rangle - \mu^2\|Bx_t - Bz\|^2\\ &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + 2\mu\|x_t - u_t\|\|Bx_t - Bz\|\\ &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + M\|Bx_t - Bz\|,\end{aligned}$$

and

$$\begin{aligned}\|y_t - z\|^2 &\le \|u_t - z\|^2 - \|u_t - y_t\|^2 + 2\lambda\langle u_t - y_t, Au_t - Az\rangle - \lambda^2\|Au_t - Az\|^2\\ &\le \|u_t - z\|^2 - \|u_t - y_t\|^2 + 2\lambda\|u_t - y_t\|\|Au_t - Az\|\\ &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 - \|u_t - y_t\|^2\\ &\quad + M\|Au_t - Az\| + M\|Bx_t - Bz\|. \end{aligned} \tag{3.6}$$

By (3.4) and (3.6), we have

$$\begin{aligned}\|x_t - z\|^2 &\le \|y_t - z\|^2 + tM\\ &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 - \|u_t - y_t\|^2\\ &\quad + M(\|Bx_t - Bz\| + \|Au_t - Az\| + t).\end{aligned}$$

It follows that

$$\|x_t - u_t\|^2 + \|u_t - y_t\|^2 \le (\|Bx_t - Bz\| + \|Au_t - Az\| + t)M.$$

This together with (3.5) imply that

$$\lim_{t\to 0}\|x_t - u_t\| = \lim_{t\to 0}\|u_t - y_t\| = 0.$$

It follows that

$$\begin{aligned}\|x_t - Sx_t\| &= \|SP_C[(1-t)y_t] - SP_C x_t\|\\ &\le (1-t)\|y_t - x_t\| + t\|x_t\|\\ &\le \|y_t - u_t\| + \|u_t - x_t\| + t\|x_t\| \to 0. \end{aligned} \tag{3.7}$$

Next we show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\} \subset (0, 1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$, $u_n := u_{t_n}$ and $y_n := y_{t_n}$. From (3.7), we get

$$\|x_n - Sx_n\| \to 0. \tag{3.8}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $x^* \in C$. Also $y_n \to x^*$ weakly. Noticing (3.8) we can use Lemma 2.4 to get $x^* \in F(S)$.

Now we show $x^* \in EP(F, B)$. Since $u_n = T_\mu(x_n - \mu Bx_n)$, for any $y \in C$ we have

$$F(u_n, y) + \frac{1}{\mu}\langle y - u_n, u_n - (x_n - \mu Bx_n)\rangle \geq 0.$$

From the monotonicity of F, we have

$$\frac{1}{\mu}\langle y - u_n, u_n - (x_n - \mu Bx_n)\rangle \geq F(y, u_n), \quad \forall y \in C.$$

Hence,

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\mu} + Bx_{n_i}\right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{3.9}$$

Put $z_t = ty + (1-t)x^*$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.9) we have

$$\begin{aligned}\langle z_t - u_{n_i}, Bz_t\rangle &\geq \langle z_t - u_{n_i}, Bz_t\rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\mu} + Bx_{n_i}\right\rangle \\ &\quad + F(z_t, u_{n_i}) \\ &= \langle z_t - u_{n_i}, Bz_t - Bu_{n_i}\rangle + \langle z_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle \\ &\quad - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\mu}\right\rangle + F(z_t, u_{n_i}). \end{aligned} \tag{3.10}$$

Note that $\|Bu_{n_i} - Bx_{n_i}\| \leq \frac{1}{\beta}\|u_{n_i} - x_{n_i}\| \to 0$. Further, from monotonicity of B, we have $\langle z_t - u_{n_i}, Bz_t - Bu_{n_i}\rangle \geq 0$. Letting $i \to \infty$ in (3.10), we have

$$\langle z_t - x^*, Bz_t\rangle \geq F(z_t, x^*). \tag{3.11}$$

From (H1), (H4) and (3.11), we also have

$$\begin{aligned} 0 = F(z_t, z_t) &\leq tF(z_t, y) + (1-t)F(z_t, x^*) \\ &\leq tF(z_t, y) + (1-t)\langle z_t - x^*, Bz_t\rangle \\ &= tF(z_t, y) + (1-t)t\langle y - x^*, Bz_t\rangle \end{aligned}$$

and hence

$$0 \leq F(z_t, y) + (1-t)\langle Bz_t, y - x^*\rangle. \tag{3.12}$$

Letting $t \to 0$ in (3.12), we have, for each $y \in C$,

$$0 \leq F(x^*, y) + \langle y - x^*, Bx^*\rangle.$$

This implies that $x^* \in EP(F, B)$. By the same argument as that of [8], we have $x^* \in VI(C, A)$. Therefore, $x^* \in \Omega$.

By (3.2), we deduce

$$\begin{aligned}\|x_t - x^*\|^2 &= \|SP_C[(1-t)y_t] - SP_C[x^*]\|^2 \\ &\le \|y_t - x^* - ty_t\|^2 \\ &= \|y_t - x^*\|^2 - 2t\langle y_t, y_t - x^*\rangle + t^2\|y_t\|^2 \\ &= \|y_t - x^*\|^2 - 2t\langle y_t - x^*, y_t - x^*\rangle - 2t\langle x^*, y_t - x^*\rangle + t^2\|y_t\|^2 \\ &\le (1-2t)\|y_t - x^*\|^2 + 2t\langle x^*, x^* - y_t\rangle + t^2\|y_t\|^2 \\ &\le 7(1-2t)\|x_t - x^*\|^2 + 2t\langle x^*, x^* - y_t\rangle + t^2\|y_t\|^2.\end{aligned}$$

It follows that

$$\|x_t - x^*\|^2 \le \langle x^*, x^* - y_t\rangle + \frac{tM}{2}.$$

In particular,

$$\|x_n - x^*\|^2 \le \langle x^*, x^* - y_n\rangle + \frac{t_n M}{2}, \quad x^* \in \Omega. \tag{3.13}$$

Hence, the weak convergence of $\{y_n\}$ to x^* implies that $x_n \to x^*$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0$.

Now we return to (3.13) and take the limit as $n \to \infty$ to get

$$\|x^* - z\|^2 \le \langle z, z - x^*\rangle, \quad z \in \Omega. \tag{3.14}$$

To show that the entire net $\{x_t\}$ converges to x^*, assume $x_{s_n} \to \tilde{x} \in \Omega$, where $s_n \to 0$. In (3.14), we take $z = \tilde{x}$ to get

$$\|x^* - \tilde{x}\|^2 \le \langle \tilde{x}, \tilde{x} - x^*\rangle. \tag{3.15}$$

Interchange x^* and $\tilde{x}$ to obtain

$$\|\tilde{x} - x^*\|^2 \le \langle x^*, x^* - \tilde{x}\rangle. \tag{3.16}$$

Adding up (3.15) and (3.16) yields

$$2\|x^* - \tilde{x}\|^2 \le \|x^* - \tilde{x}\|^2,$$

which implies that $\tilde{x} = x^*$.

We note that (3.14) is equivalent to

$$\|x^*\|^2 \le \langle x^*, z\rangle, \quad z \in \Omega.$$

This clearly implies that

$$\|x^*\| \le \|z\|, \quad z \in \Omega.$$

Therefore, x^* solves the quadratic minimization problem (3.1). This completes the proof. □

Next we introduce an explicit algorithm for finding a solution of the quadratic minimization problem (3.1). This scheme is obtained by discretizing the implicit scheme (3.2). We will show the strong convergence of this algorithm.

Theorem 3.2 *Suppose that $\Omega \neq \emptyset$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S P_C[(1-\alpha_n) P_C(I-\lambda A) T_\mu (I-\mu B) x_n], \quad n \geq 0, \quad (3.17)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to a solution of the minimization problem (3.1).

Proof First, we prove that the sequence $\{x_n\}$ is bounded.

Let $z \in \Omega$. Set $u_n = T_\mu(x_n - \mu B x_n)$ and $y_n = P_C(I-\lambda A) u_n$ for all $n \geq 0$. From (3.17), we get

$$\begin{aligned} \|y_n - z\| &= \|P_C(I-\lambda A)u_n - P_C(I-\lambda A)z)\| \\ &\leq \|u_n - z\| \\ &= \|T_\mu(x_n - \mu B x_n) - T_\mu(z - \mu B z)\| \\ &\leq \|x_n - z\|, \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - z\| &= \|\beta_n(x_n - z) + (1-\beta_n)(SP_C[(1-\alpha_n)y_n] - z)\| \\ &\leq \beta_n \|x_n - z\| + (1-\beta_n)\|(1-\alpha_n)(y_n - z) - \alpha_n z\| \\ &\leq \beta_n \|x_n - z\| + (1-\beta_n)[(1-\alpha_n)\|x_n - z\| + \alpha_n \|z\|] \\ &= [1-(1-\beta_n)\alpha_n]\|x_n - z\| + \alpha_n(1-\beta_n)\|z\| \\ &\leq \max\{\|x_n - z\|, \|z\|\}. \end{aligned}$$

By induction, we obtain, for all $n \geq 0$,

$$\|x_n - z\| \leq \max\{\|x_0 - z\|, \|z\|\}.$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce that $\{u_n\}$ and $\{y_n\}$ are all bounded. Let $M > 0$ be a constant such that

$$\sup_n \left\{\|y_n\|, 2\|y_n\|\|y_n - z\| + \|u_n\|^2, 2\|x_n - z\|, \mu\|x_n - u_n\|, 2\lambda\|u_n - y_n\|\right\} \leq M.$$

Next we show $\lim_{n\to\infty} \|y_n - S y_n\| = 0$.

Define $x_{n+1} = \beta_n x_n + (1-\beta_n) v_n$ for all $n \geq 0$. It follows from (3.17) that

$$\begin{aligned} \|v_{n+1} - v_n\| &= \|SP_C[(1-\alpha_{n+1})y_{n+1}] - SP_C[(1-\alpha_n)y_n]\| \\ &\leq \|(1-\alpha_{n+1})y_{n+1} - (1-\alpha_n)y_n\| \\ &\leq \|y_{n+1} - y_n\| + \alpha_{n+1}\|y_{n+1}\| + \alpha_n\|y_n\| \\ &\leq \|P_C(u_{n+1} - \lambda A u_{n+1}) - P_C(u_n - \lambda A u_n)\| + M(\alpha_{n+1} + \alpha_n) \\ &\leq \|u_{n+1} - u_n\| + M(\alpha_{n+1} + \alpha_n) \\ &= \|T_\mu(x_{n+1} - \mu B x_{n+1}) - T_\mu(x_n - \mu B x_n)\| + M(\alpha_{n+1} + \alpha_n) \\ &\leq \|x_{n+1} - x_n\| + M(\alpha_{n+1} + \alpha_n). \end{aligned}$$

This together with (i) imply that

$$\limsup_{n\to\infty} (\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence by Lemma 2.3, we get

$$\lim_{n\to\infty} \|v_n - x_n\| = 0.$$

It follows that

$$\begin{aligned}\|Sy_n - x_n\| &\le \|Sy_n - v_n\| + \|v_n - x_n\| \\ &= \|SP_C y_n - SP_C[(1-\alpha_n)y_n]\| + \|v_n - x_n\| \\ &\le \alpha_n \|y_n\| + \|v_n - x_n\| \to 0.\end{aligned}$$

At the same time, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|v_n - x_n\| = 0.$$

By the convexity of the norm $\|\cdot\|$, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\beta_n(x_n - z) + (1-\beta_n)(v_n - z)\|^2 \\ &\le \beta_n\|x_n - z\|^2 + (1-\beta_n)\|v_n - z\|^2 \\ &\le \beta_n\|x_n - z\|^2 + (1-\beta_n)\|y_n - z - \alpha_n y_n\|^2 \\ &= \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|y_n - z\|^2 - 2\alpha_n\langle y_n, y_n - z\rangle \\ &\quad + \alpha_n^2\|y_n\|^2] \\ &\le \beta_n\|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 + \alpha_n M. \end{aligned} \tag{3.18}$$

Since T_μ and P_C are nonexpansive and A, B are α-inverse-strongly monotone and β-inverse-strongly monotone, we have from Lemma 2.2 that

$$\begin{aligned}\|y_n - z\|^2 &= \|P_C(I - \lambda A)u_n - P_C(z - \lambda Az)\|^2 \\ &\le \|u_n - \lambda Au_n - (z - \lambda Az)\|^2 \\ &\le \|u_n - z\|^2 + \lambda(\lambda - 2\alpha)\|Au_n - Az\|^2.\end{aligned}$$

and

$$\begin{aligned}\|u_n - z\|^2 &= \|T_\mu(x_n - \mu Bx_n) - T_\mu(z - \mu Bz)\|^2 \\ &\le \|(x_n - \mu Bx_n) - (z - \mu Bz)\|^2 \\ &\le \|x_n - z\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bz\|^2.\end{aligned}$$

So, we have that

$$\|y_n - z\|^2 \le \|x_n - z\|^2 + \lambda(\lambda - 2\alpha)\|Au_n - Az\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bz\|^2. \tag{3.19}$$

Substituting (3.19) into (3.18), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|x_n - z\|^2 + \lambda(\lambda - 2\alpha)\|Au_n - Az\|^2 \\ &\quad + \mu(\mu - 2\beta)\|Bx_n - Bz\|^2] + \alpha_n M \\ &= \|x_n - z\|^2 + (1-\beta_n)\lambda(\lambda - 2\alpha)\|Au_n - Az\|^2 \\ &\quad + (1-\beta_n)\mu(\mu - 2\beta)\|Bx_n - Bz\|^2 + \alpha_n M.\end{aligned}$$

Therefore,

$$\begin{aligned}
&(1-\beta_n)\lambda(2\alpha-\lambda)\|Au_n - Az\|^2 + (1-\beta_n)\mu(2\beta-\mu)\|Bx_n - Bz\|^2 \\
&\quad \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M \\
&\quad \le (\|x_n - z\| + \|x_{n+1} - z\|)\|x_n - x_{n+1}\| + \alpha_n M \\
&\quad \le M(\|x_n - x_{n+1}\| + \alpha_n).
\end{aligned}$$

Since $\liminf_{n\to\infty}(1-\beta_n)\lambda(2\alpha-\lambda) > 0, \liminf_{n\to\infty}(1-\beta_n)\mu(2\beta-\mu) > 0, \|x_n - x_{n+1}\| \to 0$ and $\alpha_n \to 0$, we derive

$$\lim_{n\to\infty} \|Au_n - Az\| = \lim_{n\to\infty} \|Bx_n - Bz\| = 0.$$

From Lemma 2.1 and 2.2, we obtain

$$\begin{aligned}
\|u_n - z\|^2 &= \|T_\mu(x_n - \mu Bx_n) - T_\mu(z - \mu Bz)\|^2 \\
&\le \langle (x_n - \mu Bx_n) - (z - \mu Bz), u_n - z\rangle \\
&= \frac{1}{2}\big(\|(x_n - \mu Bx_n) - (z - \mu Bz)\|^2 + \|u_n - z\|^2 \\
&\quad - \|(x_n - z) - \mu(Bx_n - Bz) - (u_n - z)\|^2\big) \\
&\le \frac{1}{2}\big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|(x_n - u_n) - \mu(Bx_n - Bz)\|^2\big) \\
&= \frac{1}{2}\big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2\mu\langle x_n - u_n, Bx_n - Bz\rangle - \mu^2\|Bx_n - Bz\|^2\big),
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - z\|^2 &= \|P_C(u_n - \lambda Au_n) - P_C(z - \lambda Az)\|^2 \\
&\le \langle (u_n - \lambda Au_n) - (z - \lambda Az), y_n - z\rangle \\
&= \frac{1}{2}\big(\|(u_n - \lambda Au_n) - (z - \lambda Az)\|^2 + \|y_n - z\|^2 \\
&\quad - \|(u_n - \lambda Au_n) - (z - \lambda Az) - (y_n - z)\|^2\big) \\
&\le \frac{1}{2}\big(\|u_n - z\|^2 + \|y_n - z\|^2 - \|(u_n - y_n) - \lambda(Au_n - Az)\|^2\big) \\
&= \frac{1}{2}\big(\|u_n - z\|^2 + \|y_n - z\|^2 - \|u_n - y_n\|^2 \\
&\quad + 2\lambda\langle u_n - y_n, Au_n - Az\rangle - \lambda^2\|Au_n - Az\|^2\big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|u_n - z\|^2 &\le \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2\mu\langle x_n - u_n, Bx_n - Bz\rangle - \mu^2\|Bx_n - Bz\|^2 \\
&\le \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2\mu\|x_n - u_n\|\|Bx_n - Bz\| \\
&\le \|x_n - z\|^2 - \|x_n - u_n\|^2 + M\|Bx_n - Bz\|, \qquad (3.20)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - z\|^2 &\le \|u_n - z\|^2 - \|u_n - y_n\|^2 + 2\lambda\langle u_n - y_n, Au_n - Az\rangle - \lambda^2\|Au_n - Az\|^2 \\
&\le \|u_n - z\|^2 - \|u_n - y_n\|^2 + 2\lambda\|u_n - y_n\|\|Au_n - Az\| \\
&\le \|u_n - z\|^2 - \|u_n - y_n\|^2 + M\|Au_n - Az\|. \qquad (3.21)
\end{aligned}$$

By (3.18) and (3.20), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 + \alpha_n M \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\|u_n - z\|^2 + \alpha_n M \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n)[\|x_n - z\|^2 - \|x_n - u_n\|^2 \\
&\quad + M\|Bx_n - Bz\|] + \alpha_n M \\
&\le \|x_n - z\|^2 - (1-\beta_n)\|x_n - u_n\|^2 + (\|Bx_n - Bz\| + \alpha_n)M.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(1-\beta_n)\|x_n - u_n\|^2 &\le (\|x_n - z\| - \|x_{n+1} - z\|)(\|x_n - z\| + \|x_{n+1} - z\|) \\
&\quad + (\|Bx_n - Bz\| + \alpha_n)M \\
&\le (\|x_{n+1} - x_n\| + \|Bx_n - Bz\| + \alpha_n)M.
\end{aligned}
$$

Since $\liminf_{n\to\infty}(1-\beta_n) > 0$, $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|Bx_n - Bz\| \to 0$, we derive that

$$
\lim_{n\to\infty} \|x_n - u_n\| = 0.
$$

By (3.18) and (3.21), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 + \alpha_n M \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n)[\|u_n - z\|^2 - \|u_n - y_n\|^2 \\
&\quad + M\|Au_n - Az\|] + \alpha_n M \\
&\le \|x_n - z\|^2 - (1-\beta_n)\|u_n - y_n\|^2 + (\|Au_n - Az\| + \alpha_n)M.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(1-\beta_n)\|u_n - y_n\|^2 &\le (\|x_n - z\| - \|x_{n+1} - z\|)(\|x_n - z\| + \|x_{n+1} - z\|) \\
&\quad +(\|Au_n - Az\| + \alpha_n)M \\
&\le (\|x_{n+1} - x_n\| + \|Au_n - Az\| + \alpha_n)M.
\end{aligned}
$$

Since $\liminf_{n\to\infty}(1-\beta_n) > 0$, $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|Au_n - Az\| \to 0$, we derive that

$$
\lim_{n\to\infty} \|u_n - y_n\| = 0.
$$

Hence,

$$
\|Sy_n - y_n\| \le \|Sy_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\| \to 0.
$$

Let the net $\{x_t\}$ be defined by (3.2). By Theorem 3.1, we have $x_t \to x^* = P_\Omega(0)$ as $t \to 0$. Next we prove

$$
\limsup_{n\to\infty} \langle x^*, x^* - y_n \rangle \le 0.
$$

Indeed, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$
\limsup_{n\to\infty} \langle x^*, x^* - y_n \rangle = \lim_{i\to\infty} \langle x^*, x^* - y_{n_i} \rangle.
$$

Without loss of generality, we may further assume that $y_{n_i} \to \tilde{x}$ weakly. By the same argument as that of Theorem 3.1, we can deduce that $\tilde{x} \in \Omega$. Therefore, by using (2.1), we get

$$\limsup_{n\to\infty} \langle x^*, x^* - y_n \rangle = \langle x^*, x^* - \tilde{x} \rangle \le 0.$$

From (3.17), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\|(1-\alpha_n)(y_n - x^*) - \alpha_n x^*\|^2 \\
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)[(1-\alpha_n)^2\|y_n - x^*\|^2 \\
&\quad - 2\alpha_n(1-\alpha_n)\langle x^*, y_n - x^*\rangle + \alpha_n^2\|x^*\|^2] \\
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)[(1-\alpha_n)^2\|x_n - x^*\|^2 \\
&\quad - 2\alpha_n(1-\alpha_n)\langle x^*, y_n - x^*\rangle + \alpha_n^2\|x^*\|^2] \\
&\le [1 - 2(1-\beta_n)\alpha_n]\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle x^*, x^* - y_n\rangle + (1-\beta_n)\alpha_n^2 M \\
&= (1-\gamma_n)\|x_n - x^*\|^2 + \delta_n\gamma_n,
\end{aligned}$$

where $\gamma_n = 2(1-\beta_n)\alpha_n$ and $\delta_n = (1-\alpha_n)\langle x^*, x^* - y_n\rangle + \frac{\alpha_n M}{2}$. It is clear that $\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta \le 0$. Hence, all conditions of Lemma 2.5 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$. This completes the proof. □

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